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Set theory. With an introduction to real point sets. (English)
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The book under review grew out of an undergraduate course in introductory set theory that the author taught at the University of Detroit Mercy, USA. In contrast to many of the numerous modern texts on the subject, which introduce specific formal axiom systems for set theory in the very beginning, the present undergraduate textbook develops the core material on cardinals, ordinals, and the real line $\mathbb{R}$ in an informal, predominantly intuitive but nevertheless concrete and rigorous manner. As the author points out in the preface, this didactically motivated approach is to avoid getting bogged down in the metamathematical details of a strictly formal development of the foundations of set theory, on the one hand, and to make this text suitable for a wide range of students on the other. However, interested students will find an introduction to axiomatic Zemelo-Fraenkel set theory as well as an outlook to further landmarks of modern set theory in the last part of the book, so to speak as an optional extra justifying the informal approach presented before.

Due to this particular organization of the material, the prerequisites for the core material of the text are confined to purely undergraduate courses in mathematics, including some familiarity with mathematical reasoning and proving.

As for the precise contents, the material is divided into four main parts, whose relative independence allows some flexibility regarding topic selection for both self-reliant reading and teaching courses, respectively.

Chapter 1 is of preliminary nature and gives an informal review of naive sets, relations, families, linear orders, and functions, thus compiling prerequisite material for the rest of the book.

Part I comes with the title "Dedekind: Numbers" and comprises the subsequent Chapters 2–4, the primary goal of which is to describe the construction of real numbers. Chapter 2 derives the natural numbers from the Dedekind-Peano axioms, then develops the positive rationals out of the naturals, and ends with a discussion of Dedekind's principle of recursive definition. Chapter 3 gives the construction of the real numbers via the method of Dedekind outs, along with the underlying central idea of a general linearly ordered continuum and its fundamental properties. Chapter 4 is actually a postscript to Part I, basically consisting of philosophical and historical remarks concerning the nature of the natural numbers, ranging from the absolutist viewpoint (Russell, Zermelo, Frege, von Neumann) to Dedekind's structuralist approach as described in Chapter 2.

Part II contains the Chapters 5–12 and may be regarded as the core of the book. The headline of this part is "Cantor: Cardinals, Order, and Ordinals", and its principal goal is to present the according Cantor-Dedekind theory in an informal and naive (i.e., non-axiomatic), but still mathematically rigorous way.

Chapter 5 covers the basics of cardinal numbers, comparability, finite sets, infinite sets, countability, uncoutability, the axiom of choice, and effectiveness in this context. Chapter 6 continues the basic theory of cardinal numbers, with particular emphasis on cardinal arithmetic, the Cantor-Schröder-Bernstein theorem, binary trees, the construction of the Cantor set, and further related fundmentals. Chapter 7 briefly introduces orders, order types, and the basic operations of the latter, whereas Chapter 8 deals with some topological concepts in the context of orders. The theories of dense orders and complete orders are developed in full generality, enhanced by such topics as Cantor's theorem on countable dense linear orders, Dedekind completeness and completions, connectedness and intermediate value theorems for linear continuums, and the perfect set theorem for complete orders.

Chapter 9 turns to the classical theory of well-orders and ordinals in their naive setting, thereby defining ordinals as order types of well-orders. The discussion here includes the basic operations with ordinals, transfinite induction and recursion, the comparability theorem for well-orders, and further ordinal arithmetic.

Chapter 10 introduces countable and uncountable ordinals, the alephs and their basic arithmetic, initial ordinals and Hartogs' theorem, Zermelo's well-ordering theorem vs. the axiom of choice, the comparability for cardinals, the concept of cofinality, and the (generalized) continuum hypothesis, among other related topics concerning cardinals and ordinals.

Chapter 11 is devoted to the basics of the following complementary topics: Partial orders, Zorn's lemma, rank functions, trees, König's infinity lemma, Ramsey's theorem on infinite sets, and further combinatorial aspects of set theory.

Chapter 12, the postscript to Part II, indicates how the results of the previous chapter lead to the more advanced area of infinitary combinatorics, with special emphasis on weakly compact cardinals, Suslin's hypothesis, Martin's axiom, and Jensen's diamond principle. Part III of the book titled "Real Point Sets", illustrates how the theory of sets and orders is intimately related to the continuum and its topology. Comprising the Chapters 13–19, this part touches upon the rudiments of descriptive set theory and general topology, with the didactic focus on gaining intuition: rather than on axiomatic formality.

Chapter 13 uses the nested intervals theorem to construct interval trees, Cantor systems, and generalized Cantor sets, while Chapter 14 explains the basic (Euclidean) topology of the real line, including continuous functions, homeomorphisms, and space filling Peano curves. Chapter 15 covers the Heine-Borel theorem, sets of Lebesgue measure zero and Lebesgue measurable sets, the Baire category theorem as well as Vitali and Bernstein sets in $\mathbb{R}$.

Chapter 16 is more special in that it discusses the Cantor-Bendixson analysis of countable closed sets, together with an application to Cantor's uniqueness theorem for trigonometric series.

Chapter 17 gives an application of the theory of orders (from Chapter 8) to prove two further classical theorems, namely Brouwer's topological characterization of the Cantor set and Sierpiński's topological description of the set of rational numbers.

Chapter 18 treats some of the basic theory of sigma-algebras, Borel sets, Suslin systems, and analytic sets in $\mathbb{R}$. As for the latter, the perfect set property, measurability, and the Baire property are shown, and the Lusin separation theorem, Suslin's theorem for

analytic sets, the boundedness theorem, and an example of a non-Borel analytic set are presented along the way.
Chapter 19, the postscript to Part III, describes two classical problems of real analysis beyond the usual axiomatic set theory:
The so-called Lebesgue measure extension problem and approaches by Vitali-Bernstain, Banach-Ulam, Tarski-Ulam, and Ulam himself, on the one hand, and Lusin's problem on properties on projective sets on the other.
Finally, Part IV gives an overview of the formal-logical and purely foundational issues of set theory. Chapter 20 presents some of celebrated classical paradoxes of naive set theory (as having been used in the previous parts of the book) and the attempts of their resolution by Frege, Russell, Zermelo, Skolem, and others.
Chapter 21 describes the Zermelo-Fraenkel axiom system for a set theory, J. von Neumann's, definition of ordinals, the axiom of infinity, Cantor-von Neumann cardinals, the idea of the von Neumann-Bernays set theory, and the philosophy of W. V. O. Quine's new foundations of logic. In Chapter 22, the postscript to Part IV, further landmarks of modern set theory are briefly sketched, in particular Gödel's axiom of constructibility, Cohen's method of forcing, large cardinal axioms (à la Solovay), infinite games, principles of indeterminacy, projective games, projective indeterminacy, and the present status of the Continuum Hypothesis.
The main text is supplemented by three appendices. Appendix A presents three different proofs of the uncountability of $\mathbb{R}$, Appendix B gives a proof of the existence of the Lebesgue measure on $\mathbb{R}$, and Appendix C provides a list of the Zermelo-Fraenkel axioms for the reader's convenience.
In general, the whole book is largely problem-based. The numerous problems form an essential part of the author's teaching strategy, and they are basically meant to provide an extension of the main text. The harder problems are equipped with useful hints or even outlines for solution.
All together, this lucidly written undergraduate set theory textbook is a welcome addition to the relevant literature, with many individual features and a remarkably high degree of thematic versatility.

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Keywords : set theory (textbook); cardinal numbers; ordinal numbers; ordered sets; combinatorial set theory; Zermelo-Fraenkel axioms; von Neumann ordinals; large cardinals; number systems; descriptive set theory; Cantor sets
Classification :
∗03-01 Textbooks (mathematical logic)
00A05 General mathematics
03Exx Set theory (logic)
26A03 Elementary topology of the real line
28A05 Classes of sets
54H05 Descriptive set theory (topological aspects)