2.10 Recursive Definitions*

Recall that we had “defined” addition of natural numbers by the following *recursion equations*:

$$m+1 := S(m), \qquad \text{and} \qquad m+S(n) := S(m+n).$$

But this is not an explicit definition! We took it for granted (as was done in the work of Peano) that a two-place function + (the mapping $(m,n) \mapsto m+n$) satisfying the above equations exists, without giving any rigorous justification for its existence. Similarly, multiplication of natural numbers was “defined” by recursion equations without proper justification.

Dedekind introduced a general method, known as *primitive recursion,* which provides such justification. It assures the *existence and uniqueness* of functions which are defined implicitly using recursion equations having forms similar to the ones for addition and multiplication.

We will formulate and prove a general version of Dedekind’s principle of recursive definition, from which the existence and uniqueness for the sum and product functions can be immediately derived.

Principles of Recursive Definition

The following *Basic Principle of Recursive Definition* is perhaps the simplest yet very useful result for defining functions recursively.

Theorem 146 (Basic Principle of Recursive Definition). *If Y is a set, $a \in Y$, and $h\colon Y \to Y$, then there is a unique $f\colon \mathbf{N} \to Y$ such that*

$$f(1) = a, \qquad \textit{and} \qquad f(S(n)) = h(f(n)) \ \textit{for all } n \in \mathbf{N}.$$

Informally, this says that given $a \in Y$ and $h\colon Y \to Y$, we can form the infinite sequence $\langle a, h(a), h(h(a)), \ldots \rangle$.

Proof. The uniqueness of the function f can be established by an easy and routine induction, so let us prove existence.

A subset I of $\mathbf{N}$ will be called an *initial set* if for all $k \in \mathbf{N}$, if $S(k) \in I$ then $k \in I$. By routine induction, we can establish the following:

- $\{1\}$ is an initial set, and every non-empty initial set contains 1 as a member.
- If I is an initial set with $k \in I$ then $I \cup \{S(k)\}$ is also an initial set.
- For each $n \in \mathbf{N}$, there is a unique initial set I such that $n \in I$ but $S(n) \notin I$.

Let I_n denote the unique initial set containing n but not $S(n)$. It follows that $I_1 = \{1\}$, and $I_{S(n)} = I_n \cup \{S(n)\}$ for all $n \in \mathbf{N}$. (Informally, $I_n = \{1,2,\ldots,n\}$, the set of first n natural numbers.) The proof will use *functions $u\colon I_n \to Y$ having domain I_n.*

Let us say that a function *u is partially h-recursive with domain I_n* if $u\colon I_n \to Y$, $u(1) = a$, and $u(S(k)) = h(u(k))$ for all k such that $S(k) \in I_n$.

We first prove by induction that for every $n \in \mathbf{N}$ there is a unique partially h-recursive u with domain I_n.

Basis step ($n = 1$): Let $v: \{1\} \to Y$ be defined by setting $v(1) = a$. Then v is partially h-recursive with domain I_1. Moreover, if $u, u': I_1 \to Y$ are partially h-recursive functions with domain I_1, then $u(1) = a = u'(1)$, so $u = u'$ since 1 is the only element in their domain $I_1 = \{1\}$. So there is a unique partially h-recursive v with domain I_1, establishing the basis step.

Induction step: Suppose that $n \in \mathbf{N}$ is such that there is a unique partially h-recursive v with domain I_n (induction hypothesis). We fix this v for the rest of this step, and define $w: I_{S(n)} \to Y$ by setting $w(k) := v(k)$ for $k \in I_n$ and $w(k) := h(v(n))$ if $k = S(n)$. Then w is easily seen to be partially h-recursive with domain $I_{S(n)}$. Moreover, if $u, u': I_{S(n)} \to Y$ are partially h-recursive with domain $I_{S(n)}$, then the restrictions $u \restriction_{I_n}$ and $u' \restriction_{I_n}$ are partially h-recursive with domain I_n, so they must be identical by induction hypothesis, i.e. $u(k) = u'(k)$ for $k \in I_n$. In particular, $u(n) = u'(n)$, so $u(S(n)) = h(u(n)) = h(u'(n)) = u'(S(n))$, which gives $u = u'$. Hence there is a unique partially h-recursive w with domain $I_{S(n)}$, which finishes the induction step.

Thus for each n there is a unique partially h-recursive function with domain I_n; let us denote this function by u_n.

Now define $f: \mathbf{N} \to Y$ by setting:

$$f(n) := u_n(n).$$

First, $f(1) = a$ since $u_1(1) = a$. Next, the restriction of $u_{S(n)}$ to I_n equals u_n (by uniqueness, since the restriction is partially h-recursive), so $u_{S(n)}(n) = u_n(n)$. Hence $f(S(n)) = u_{S(n)}(S(n)) = h(u_{S(n)}(n)) = h(u_n(n)) = h(f(n))$. Thus f satisfies the recursion equations of the theorem. □

To handle functions of multiple variables, the following theorem is used.

Theorem 147 (General Principle of Recursive Definition). *For any $g: X \to Y$ and $h: X \times \mathbf{N} \times Y \to Y$, there is a unique function $f: X \times \mathbf{N} \to Y$ such that for all $x \in X$ and $n \in \mathbf{N}$:*

$$f(x,1) = g(x) \quad \textit{and} \quad f(x,S(n)) = h(x,n,f(x,n)).$$

Here f is being defined by recursion on the second variable n, that is, n is the *variable of recursion* ranging over $\mathbf{N}$, while x is a *parameter* ranging over the set X. This is the most general form of recursive definition, where both the parameters (in X) and the values (in Y) come from arbitrary sets.

Proof. The proof is essentially the same as that of Theorem 146, since the additional parameter does not play any significant role in the recursion. The details are left as an exercise for the reader. □

Theorem 148 (Course of Values Recursion). *Let Y be a non-empty set and Y^* denote the set of all finite sequences (strings) of elements from Y. Given any $G: Y^* \to Y$ there is a unique $f: \mathbf{N} \to Y$ such that*