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GENERAL TOPOLOGY

Compactness and symmetric well-orders

by

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Summary. We introduce and investigate a topological form of Stäckel's 1907 characterization of finite sets, with the goal of obtaining an interesting notion that characterizes usual compactness (or a close variant of it). Define a T_2 topological space (X,τ) to be *Stäckel-compact* if there is some linear ordering $\prec$ on X such that every non-empty τ-closed set contains a $\prec$-least and a $\prec$-greatest element. We find that compact spaces are Stäckel-compact but not conversely, and Stäckel-compact spaces are countably compact. The equivalence of Stäckel-compactness with countable compactness remains open, but our main result is that this equivalence holds in scattered spaces of Cantor–Bendixson rank $< \omega_2$ under ZFC. Under $V = L$, the equivalence holds in all scattered spaces.

1. Introduction and summary of results. A familiar phenomenon in point-set topology is that a purely combinatorial set-theoretic condition that characterizes finiteness of sets will often have a corresponding analogue in topological spaces which characterizes compactness, or at least an interesting variant of compactness (Tao [8] illustrates this with example properties; see also [1]).

The purpose of this article is to introduce and investigate the topological analogue of a specific property due to Stäckel [6] that characterizes the finiteness of a set, namely the existence of some ordering on the set in which every non-empty subset has a smallest and a greatest element (a "symmetric well-order"). Section 2 defines the corresponding topological property which we call *Stäckel-compactness*, namely the existence of some ordering on the space such that every non-empty *closed* subset has a smallest and a great-

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est element. Our problem is to study how close this notion is to ordinary compactness.

In Section 3, we establish some basic properties of Stäckel-compactness: We observe that every compact Hausdorff space is Stäckel-compact, but not conversely (we show that the space ω_1 is Stäckel-compact). Also, every Stäckel-compact space is countably compact. Consequently, in metric spaces Stäckel-compactness coincides with usual compactness. Thus the notion has similarities with other variants of compactness such as pseudocompactness and sequential compactness (although is distinct from them). Stäckel-compactness appears to be quite close to countable compactness, but we do not know if the two notions are distinct or if they coincide.

In Section 4, we obtain our main result: *In scattered Hausdorff spaces of Cantor–Bendixson rank less than ω_2, Stäckel-compactness coincides with countable compactness* (Theorem 4.4).

In Section 5 we go beyond ZFC and combine our results with those from [5] to remove the restriction on Cantor–Bendixson rank: *Under V=L, all scattered countably compact T_2 spaces are Stäckel-compact, so this conclusion is relatively consistent with ZFC* (Theorem 5.3). The referee has pointed out that there is a class of scattered spaces, namely Mrówka spaces, which provides examples distinguishing different notions of compactness, so Theorem 5.3 tightens the problem of distinguishing Stäckel-compactness from countable compactness under ZFC, and raises the possibility that Theorem 5.3 may hold in ZFC without additional set-theoretic axioms.

Section 6 uses a suggestion from the referee to show that Novák's basic method [3] produces Stäckel-compact spaces (Proposition 6.5), and Stäckel-compactness is not a productive property (Corollary 6.6).

Finally, in Section 7 we finish with open questions, credits, and history.

2. Symmetric well-orders and Stäckel-compactness

Definition 2.1. An order $\prec$ on a set X is called a *symmetric well-order* if every non-empty subset of X contains both a least and a greatest element (or equivalently, if both $\prec$ and the reverse order $\succ$ well order X).

Proposition 2.2 (Stäckel [6]). *A set X is finite if and only if a symmetric well-order can be defined on X.*

(This is easily proved in ZF without AC; see [7, pp. 108 and 149].)

From this purely combinatorial characterization of finite sets, we now go to its topological analogue: We define a topological version of symmetric well-ordering, and formulate our main notion, *Stäckel-compactness.*

Definition 2.3. Let X be a Hausdorff topological space.

(1) An order $\prec$ on X is called a *symmetric topological well-order* if each non-empty closed subset of X has a least and a greatest element.
(2) X is called *Stäckel-compact* if there exists some ordering $\prec$ on X which is a symmetric topological well-order.

Caveat: In these definitions, the order $\prec$ on X is not assumed to be related to the topology on X in any other way. In general, the order topology given by the order $\prec$ will be unrelated to the original topology of X.

The unit interval $[0,1]$ is Stäckel-compact, as the usual ordering is itself a symmetric topological well-order; here the topology is same as the order topology. More generally, a linearly ordered space (a linearly ordered set under order topology) is compact if and only if every non-empty closed set has a least and a greatest element, and so all compact linearly ordered spaces are Stäckel-compact. Conversely, we can ask if the points of a given topological space X can be ordered in a way that every non-empty closed set has a least and a greatest element. So we can view Stäckel-compactness as a natural generalization of compact linearly ordered spaces (rather than as a topological analogue of finiteness), which gives an alternative second motivation for our notion. I thank the referee for this observation.

More examples of Stäckel-compact spaces will appear later.

In the remaining sections, we focus on our main problem:

Problem. How close is Stäckel-compactness to compactness?

Throughout, we restrict our attention to Hausdorff topological spaces, and assume the Axiom of Choice, i.e., work in ZFC.

(Notations and results used can be found in standard references in point-set topology and set theory, such as [1, 2].)

3. Basic properties of Stäckel-compact spaces. Stäckel-compactness shares these properties of ordinary compactness:

Proposition 3.1. *A closed subspace of a Stäckel-compact space is Stäckel-compact. If a topology is Stäckel-compact, then so is any weaker T_2 topology.*

Proof. Immediate from the definition. ∎

Proposition 3.2. *Every compact Hausdorff space X is Stäckel-compact.*

Proof. X is homeomorphic to a closed subspace of $[0,1]^\mu$ for some ordinal μ. The lexicographic order on $[0,1]^\mu$ is readily verified to be a symmetric topological well-order. Hence $[0,1]^\mu$, and so X, is Stäckel-compact. ∎

Proposition 3.3. *Every Stäckel-compact space X is countably compact.*

Proof. Fix an ordering on X in which every non-empty closed set contains both a least and a greatest element. Since X is Hausdorff, it suffices to show that every infinite subset of X has a limit point. Let A be an infinite subset of X. Then (e.g. by Proposition 2.2) A contains a non-empty subset B which either has no least element or has no greatest element. B cannot be closed, since X is Stäckel-compact. Hence B has a limit point, and so A has a limit point. ∎

COROLLARY 3.4. *Let X be a Hausdorff space which is either Lindelöf or paracompact. Then X is Stäckel-compact if and only if X is compact. In particular, in metrizable spaces Stäckel-compactness coincides with compactness.*

At this point, we have the following implications (in Hausdorff spaces):

$$\text{Compact} \implies \text{Stäckel-compact} \implies \text{Countably compact}$$

We next ask: Can the first implication be reversed? Is every Stäckel-compact space compact? It turns out that the answer is negative.

THEOREM 3.5. *The space ω_1, consisting of all countable ordinals with the order topology, is Stäckel-compact. Thus there are Stäckel-compact spaces which are not compact.*

Proof. Let $<$ denote the usual ordering on ordinals. Partition ω_1 into two stationary sets A and B, and let $\prec$ be the order on ω_1 in which "A is followed by the reverse of B", or more precisely by defining $\alpha \prec \beta$ if and only if either $\alpha \in A$ and $\beta \in B$, or $\alpha, \beta \in A$ and $\alpha < \beta$, or $\alpha, \beta \in B$ and $\alpha > \beta$.

Let F be a non-empty closed subset of ω_1. We will show that F contains both a $\prec$-least element and a $\prec$-greatest element.

If F meets both A and B then F clearly has both a $\prec$-least element and a $\prec$-greatest element, so assume that either $F \subseteq A$ or $F \subseteq B$. Then F must be countable, since an uncountable closed set in ω_1 would meet both A and B. Suppose that $F \subseteq A$. Now F is a non-empty countable closed set in ω_1, and so F contains a $<$-least and a $<$-greatest ordinal under the usual ordering $<$ of the ordinals. But on A, the ordering $\prec$ coincides with $<$, so F contains both a $\prec$-least element and a $\prec$-greatest element. (A similar argument applies if $F \subseteq B$.)

Thus ω_1 is Stäckel-compact. ∎

A similar argument shows that the long line is Stäckel-compact. Also, for general ordinal spaces, it will follow from the results of this article that a limit ordinal λ of uncountable cofinality is Stäckel-compact if $\lambda < \omega_2$, and under V=L all limit ordinals of uncountable cofinality are Stäckel-compact.

The product of two Stäckel-compact spaces may not be Stäckel-compact (Corollary 6.6). However, if one of the two spaces is additionally assumed to be compact, then the product will be Stäckel-compact.

Proposition 3.6. *Let X and Y be Hausdorff spaces. If X is Stäckel-compact and Y is compact, then $X \times Y$ is Stäckel-compact.*

Proof. By Proposition 3.2, Y is also Stäckel-compact. Therefore we can fix symmetric topological well-orders on X and on Y. Then the lexicographical order on $X \times Y$ defined by

$$(u, v) \prec (x, y) \iff u \prec x \text{ in } X, \text{ or } u = x \text{ and } v \prec y \text{ in } Y$$

is a symmetric topological well-order on $X \times Y$. To see this, let C be a non-empty closed subset of $X \times Y$. Then the projection P of C onto X,

$$P := \{x \in X : (x, y) \in C \text{ for some } y \in Y\}$$

is also a non-empty closed subset of X since Y is compact. So P will have a least element, say x_0, with respect to the symmetric topological well-order on X. Now the set $Q := \{y \in Y : (x_0, y) \in C\}$ is a non-empty closed subset of Y and so will have a least element, say y_0, with respect to the symmetric topological well-order on Y. Then (x_0, y_0) will be the least element of C under the lexicographic order. Similarly C also has a largest element. ∎

Corollary 3.7. *The product space $\omega_1 \times (\omega_1 + 1)$ is Stäckel-compact. We thus have Stäckel-compact Tikhonov spaces which are not normal.* (This answers a question of T. S. S. R. K. Rao from ICAACA 2020 conference, India.)

The basic observations of this section indicate that Stäckel-compactness behaves in ways similar (but not identical) to some other variants of compactness. Like pseudocompactness and countable compactness, Stäckel-compactness is a necessary but not sufficient condition for compactness in Hausdorff spaces, and in metric spaces it coincides with compactness. Unlike pseudocompactness, Stäckel-compactness implies countable compactness. We next look at the question of reversal of this implication.

4. The case of countably compact spaces. We now ask if the second implication mentioned after Corollary 3.4 can be reversed:

Question. Are all countably compact T_2 spaces Stäckel-compact?

We do not know the full answer to this question, but we will prove a partial result and show that certain types of countably compact spaces are Stäckel compact, under certain restrictions. This is the main result of this article, Theorem 4.4.

When we try to improve Theorem 4.4 by relaxing its restrictive conditions, we get into set-theoretical considerations involving additional hypotheses beyond the standard ZFC axioms (Section 5).

However, in this section all results are proved under ZFC. First, we set up and review some standard terminology and notation.

Definition 4.1. Let X be a topological space and let E be a subset of X.

(1) $\operatorname{Lim} E$ denotes the set of limit points of E.
(2) E is *perfect* if E is closed and dense-in-itself, that is, $\operatorname{Lim} E = E$.
(3) The space X is *scattered* if no non-empty subset is perfect.

We get the *Cantor–Bendixson derivatives* of X by repeatedly applying the Lim operation through all ordinals, taking intersections at limit stages:

Definition 4.2. Let X be a topological space. For each ordinal α we define a subset $X^{(\alpha)}$ of X by transfinite recursion as follows:

$$\begin{aligned} X^{(0)} &:= X, \\ X^{(\alpha+1)} &:= \operatorname{Lim} X^{(\alpha)}, \\ X^{(\alpha)} &:= \bigcap_{\beta<\alpha} X^{(\beta)} \quad \text{if } \alpha \text{ is a limit ordinal.} \end{aligned}$$

$X^{(\alpha)}$ is called the *αth Cantor–Bendixson derivative* of X.

The Cantor–Bendixson derivatives $X^{(\alpha)}$ are closed sets that decrease with α, i.e. $\alpha < \beta \Rightarrow X^{(\alpha)} \supseteq X^{(\beta)}$:

$$X = X^{(0)} \supseteq X^{(1)} \supseteq X^{(2)} \supseteq \cdots \supseteq X^{(\alpha)} \supseteq X^{(\alpha+1)} \supseteq \cdots,$$

and there must be an ordinal ρ with $X^{(\rho+1)} = X^{(\rho)}$.

Definition 4.3. For a topological space X, the least ordinal $\rho = \rho(X)$ such that $X^{(\rho+1)} = X^{(\rho)}$ is called the *Cantor–Bendixson rank* (or *CB-rank*) of X.

Note that if $\rho = \rho(X)$ is the Cantor–Bendixson rank of X, then X is perfect if and only if $\rho = 0$, and X is scattered if and only if $X^{(\rho)} = \varnothing$.

Also, if X is countably compact and scattered, then its CB-rank $\rho = \rho(X)$ is either a successor ordinal or must have uncountable cofinality ($\operatorname{cf} \rho > \omega$).

The simplest example of a countably compact scattered Hausdorff space with uncountable CB-rank is ω_1 (under the usual order topology), which we saw in Theorem 3.5 to be Stäckel-compact. We may therefore try to somehow “lift the proof” of Theorem 3.5 to general countably compact scattered Hausdorff spaces. This is done in Theorem 4.6 below under a “reflection assumption”.

We now state our main result.

Theorem 4.4. (ZFC) *Let X be a scattered Hausdorff space with CB-rank less than ω_2. Then X is Stäckel-compact if and only if X is countably compact.*

Using Theorem 4.4, we get more examples of Stäckel-compact spaces: $X := \omega_1 \times \omega_1$, $Y := (\omega_1 + 1) \times (\omega_1 + 1) \smallsetminus \{(\omega_1, \omega_1)\}$, and $Z := \omega_1 \times (\omega_1 + 1)$.

(In Corollary 3.7, Z was shown to be Stäckel-compact using Proposition 3.6, but Proposition 3.6 does not help for X or Y.)

The rest of the section is devoted to the proof of Theorem 4.4.

Definition 4.5. Let A be a set and let α be an ordinal with $\operatorname{cf}\alpha > \omega$. We say that

(1) *A reflects at α* if $A \cap \alpha$ is stationary in α;
(2) *A reflects everywhere on ρ* (where ρ is an arbitrary ordinal) if A reflects at β for every $\beta \leq \rho$ with $\operatorname{cf}\beta > \omega$.

Theorem 4.6. (ZFC) *Let X be a countably compact scattered Hausdorff space with CB-rank $\rho = \rho(X)$, and suppose that there exist disjoint sets A and B each of which reflects everywhere on ρ. Then X is Stäckel-compact.*

Proof. The proof improves upon the proof that ω_1 is Stäckel-compact.

By the given condition, we can partition ρ into two disjoint sets A, B such that each of A and B reflects everywhere on ρ.

Define, for each ordinal α,

$$Y_\alpha := X^{(\alpha)} \setminus X^{(\alpha+1)}.$$

Then $\{Y_\alpha : \ \alpha < \rho\}$ forms a partition of X.

Fix a well-order of X such that Y_α precedes Y_β in this order if $\alpha < \beta < \rho$.

Now define

$$X_A := \bigcup_{\alpha \in A} Y_\alpha \quad \text{and} \quad X_B := \bigcup_{\beta \in B} Y_\beta.$$

Then X_A and X_B form a partition of X.

Now take the order on X in which X_A precedes X_B, X_A is ordered by the above well-order, and X_B is ordered by the reverse of that well-order.

We now show that under this order, every non-empty closed subset F of X has a least and a greatest element.

Given a non-empty closed set F in X, consider the two sets $F \cap X_A$ and $F \cap X_B$. If both of these sets are non-empty, then F will contain least and greatest elements (since X_A is well-ordered and X_B is reverse well-ordered by our new chosen order on X). So we may assume that one of the sets $F \cap X_A$ and $F \cap X_B$ is empty, and without loss of generality that $F \cap X_B = \emptyset$, that is, $F \subseteq X_A$. So F has a least element. We will show that F has a greatest element as well.

Define

$$C := \{\alpha < \rho : F \cap Y_\alpha \neq \emptyset\}.$$

Then $C \subseteq A$ by our assumption that $F \subseteq X_A$, and $C \neq \emptyset$ since $F \neq \emptyset$.

If C has a largest element μ, then $F \cap Y_\mu$ must be non-empty finite by countable compactness, and so will have a largest element, which must then be the greatest element of F. Hence it suffices to show that C has a largest element.

Suppose (for contradiction) that C does not have a largest element. Then $\sup C \in \operatorname{Lim} C \setminus C$. Let

$$\mu := \min(\operatorname{Lim} C \setminus C).$$

Thus μ is a limit ordinal $\leq \rho$, and $\operatorname{cf} \mu \geq \omega_1$ by countable compactness. Note that $C \cap \mu$ is a closed unbounded set in μ. Now, since B reflects at μ, $B \cap \mu$ is stationary in μ, and so $(C \cap \mu) \cap (B \cap \mu)$ must be non-empty. But this implies that $C \cap B \neq \emptyset$, which is a contradiction since $C \subseteq A$. ∎

Our goal now is to try to use the above theorem to show that if a scattered Hausdorff space is countably compact, then it is Stäckel-compact. But, as mentioned earlier, we are unable to do this without additional set-theoretic hypothesis beyond ZFC (Section 5). In ZFC alone, we can use the next theorem below along with Theorem 4.6 to obtain the result for spaces with CB-rank $< \omega_2$, giving us Theorem 4.4.

THEOREM 4.7. (ZFC) *If $\rho < \omega_2$, then there exist disjoint sets A and B each of which reflects everywhere on ρ.*

Proof. The proof is by induction on ρ.

Suppose that $\rho < \omega_2$ and that for every $\xi < \rho$ there are disjoint A_ξ, B_ξ each of which reflects everywhere on ξ.

Without loss of generality we can assume that ρ is a limit ordinal, so there are two cases: $\operatorname{cf} \rho = \omega$ and $\operatorname{cf} \rho = \omega_1$.

CASE 1: $\operatorname{cf} \rho = \omega$. We can then choose a countable sequence of ordinals

$$0 = \rho_0 < \rho_1 < \cdots < \rho_n < \rho_{n+1} < \cdots$$

such that $\sup_n \rho_n = \rho$. By induction hypothesis, for each $n \in \omega$ we can fix disjoint sets A_n, B_n such that both of them reflect everywhere on ρ_{n+1}. Let

$$A := \bigcup_{n\in\omega} A_n \cap (\rho_{n+1} \setminus \rho_n) \quad \text{and} \quad B := \bigcup_{n\in\omega} B_n \cap (\rho_{n+1} \setminus \rho_n).$$

(For ordinals α and β, the set-difference $\alpha \setminus \beta$ equals $\{\xi : \beta \leq \xi < \alpha\}$.)

Notice that $A \cap B = \emptyset$. We show that both A and B reflect everywhere on ρ. Suppose that $\alpha \leq \rho$ with $\operatorname{cf} \alpha \geq \omega_1$. Then $0 < \alpha < \rho$ (since $\operatorname{cf} \rho = \omega$ and $\operatorname{cf} \alpha \geq \omega_1$), and so there is n such that $\rho_n < \alpha \leq \rho_{n+1}$. Now A_n reflects everywhere on ρ_{n+1}, so $A_n \cap \alpha$ is stationary in α, and therefore $A_n \cap (\alpha \setminus \rho_n)$ is also stationary in α (as $\alpha \setminus \rho_n$ is closed unbounded in α). But

$$A_n \cap (\alpha \setminus \rho_n) \subseteq A_n \cap (\rho_{n+1} \setminus \rho_n) \subseteq A,$$

hence $A \cap \alpha$ is stationary in α. Similarly, $B \cap \alpha$ is stationary in α. Thus both A and B reflect everywhere on ρ.

CASE 2: $\operatorname{cf}\rho = \omega_1$. Fix $E \subseteq \rho$ of order type ω_1 with $\sup E = \rho$, and let $L := \rho \cap \operatorname{Lim} E$. Then L is closed unbounded in ρ of order type ω_1 and each $\lambda \in L$ is a limit ordinal of countable cofinality. Enumerate L increasingly as $\langle \lambda_\xi \rangle_{\xi<\omega_1}$:

$$L = \{\lambda_\xi : 0 \le \xi < \omega_1\} \quad \text{with } \lambda_\alpha < \lambda_\beta \text{ for all } \alpha < \beta < \omega_1.$$

Now, for each $\alpha < \omega_1$, we have $\lambda_{\alpha+1} < \rho$, so by induction hypothesis we can find disjoint sets A_α, B_α such that both A_α and B_α reflect everywhere on $\lambda_{\alpha+1}$. Define

$$A^* := \bigcup_{\alpha<\omega_1} [A_\alpha \cap (\lambda_{\alpha+1} \setminus (\lambda_\alpha + 1))],$$
$$B^* := \bigcup_{\alpha<\omega_1} [B_\alpha \cap (\lambda_{\alpha+1} \setminus (\lambda_\alpha + 1))].$$

Note that the sets A^*, B^*, and L are pairwise disjoint. As L is closed unbounded in ρ, we can fix disjoint subsets C and D of L that are stationary in ρ. Finally, define

$$A := A^* \cup C \quad \text{and} \quad B := B^* \cup D.$$

Then $A \cap B = \emptyset$. We show that both A and B reflect everywhere on ρ. Suppose that $\alpha \le \rho$ with $\operatorname{cf}\alpha > \omega$. If $\alpha = \rho$, then C, and so A, is stationary in $\rho = \alpha$. If $\alpha < \rho$, then $\lambda_\xi \le \alpha < \lambda_{\xi+1}$ for some $\xi < \omega_1$. Since the limit ordinal λ_ξ has countable cofinality but $\operatorname{cf}\alpha > \omega$, we get $\lambda_\xi < \alpha$. Also, A_ξ reflects everywhere on $\lambda_{\xi+1}$, so $A_\xi \cap \alpha$ is stationary in α, and therefore $A_\xi \cap (\alpha \setminus (\lambda_\xi + 1))$ is stationary in α (as $\alpha \setminus (\lambda_\xi + 1)$ is closed unbounded in α). But

$$A_\xi \cap (\alpha \setminus (\lambda_\xi + 1)) \subseteq A_\xi \cap (\lambda_{\xi+1} \setminus (\lambda_\xi + 1)) \subseteq A^* \subseteq A,$$

hence $A \cap \alpha$ is stationary in α. Similarly, $B \cap \alpha$ is stationary in α. Thus both A and B reflect everywhere on ρ. ∎

Theorem 4.4 now follows immediately from Theorems 4.6 and 4.7.

5. CB-rank ω_2 and beyond. This section improves Theorem 4.4 using extra set-theoretic hypotheses that are relatively consistent with ZFC. We first show that under $\square_{\omega_1}$, every countably compact scattered Hausdorff space of CB-rank ω_2 is Stäckel-compact (Corollary 5.2). Then, combining our Theorem 4.6 with a result of Hamkins [5], we get: *Under ZFC + V=L, Stäckel-compactness coincides with countable compactness in all scattered Hausdorff spaces* (Theorem 5.3).

We use set-theoretic terminology from Jech [2]. For any well-ordered set X, let $\operatorname{ord}(X)$ denote the unique ordinal order-isomorphic to X (the order-type of X). ZFC + V=L denotes the axioms of ZFC augmented with Gödel's Axiom of Constructibility V=L (which is consistent relative to ZFC).

Let κ be an uncountable cardinal. Jensen's *square principle* $\square_\kappa$ is the following statement, true under ZFC + V=L (see Jech [2]):

$\square_\kappa$ There is a sequence $\langle C_\alpha \colon \alpha < \kappa^+, \alpha \text{ a limit ordinal}\rangle$ of sets, known as a $\square_\kappa$*-sequence,* such that for all limit $\alpha < \kappa^+$,

(1) C_α is closed unbounded in α;
(2) $\operatorname{cf} \alpha < \kappa$ implies that C_α has cardinality less than κ;
(3) $\beta \in \operatorname{Lim}(C_\alpha)$ implies $C_\beta = C_\alpha \cap \beta$.

For such a $\square_\kappa$-sequence, we have $\operatorname{ord}(C_\alpha) \leq \kappa$ for all limit $\alpha < \kappa^+$.

PROPOSITION 5.1. *Assume $\square_{\omega_1}$. Then ω_2 has a pair of disjoint subsets which reflect everywhere on ω_2.*

Proof. By $\square_{\omega_1}$, we can fix a sequence $\langle C_\alpha \colon \alpha < \omega_2, \alpha \text{ a limit ordinal}\rangle$ such that for all limit $\alpha < \omega_2$,

(1) C_α is closed unbounded in α;
(2) $\operatorname{cf} \alpha = \omega_1$ implies $\operatorname{ord}(C_\alpha) = \omega_1$;
(3) $\beta \in \operatorname{Lim}(C_\alpha)$ implies $C_\beta = C_\alpha \cap \beta$.

Fix disjoint stationary subsets A_0, B_0 of ω_1 consisting of limit ordinals. For each $\mu < \omega_2$ with $\operatorname{cf} \mu = \omega_1$, the set C_μ is closed unbounded in μ and of order type ω_1, so we can form "isomorphic copies of A_0 and B_0 within C_μ" (under the unique order-isomorphism between ω_1 and C_μ) by defining

$$A_\mu := \{\xi \in C_\mu \colon \operatorname{ord}(C_\mu \cap \xi) \in A_0\} \text{ and } B_\mu := \{\xi \in C_\mu \colon \operatorname{ord}(C_\mu \cap \xi) \in B_0\}.$$

Note that $A_\mu \subseteq \operatorname{Lim} C_\mu$ and $B_\mu \subseteq \operatorname{Lim} C_\mu$. Finally, "put them all together" by defining

$$A := \bigcup_{\substack{\mu<\omega_2 \\ \operatorname{cf}\mu=\omega_1}} A_\mu \quad \text{and} \quad B := \bigcup_{\substack{\mu<\omega_2 \\ \operatorname{cf}\mu=\omega_1}} B_\mu.$$

Then A and B are disjoint, since if $\xi \in A_\mu \cap B_\nu$ then $\xi \in \operatorname{Lim} C_\mu \cap \operatorname{Lim} C_\nu$, so $C_\mu \cap \xi = C_\xi = C_\nu \cap \xi$, so $\operatorname{ord}(C_\mu \cap \xi) = \operatorname{ord}(C_\nu \cap \xi)$, which is a contradiction since $\operatorname{ord}(C_\mu \cap \xi) \in A_0$ and $\operatorname{ord}(C_\nu \cap \xi) \in B_0$, while A_0 and B_0 are disjoint.

Now if $\mu < \omega_2$ and $\operatorname{cf} \mu = \omega_1$, then A_μ and B_μ are stationary in C_μ and hence in μ, so A and B reflect at μ. Also, A and B reflect at $\mu = \omega_2$ as well, since they are stationary in ω_2. So A and B are disjoint sets which reflect everywhere on ω_2. ∎

Combining the above proposition with Theorem 4.6 we get:

COROLLARY 5.2. *Assuming $\square_{\omega_1}$, every countably compact scattered Hausdorff space of CB-rank ω_2 is Stäckel-compact.*

Hamkins (MathOverflow post [5], enclosed below) shows that if the global square principle is assumed, then for every $\rho \geq \omega_2$ there are disjoint stationary sets which reflect everywhere on ρ (in fact, there exist disjoint proper classes A, B of ordinals such that $A \cap \alpha$ and $B \cap \alpha$ are stationary in α for every ordinal α with uncountable cofinality). Since the global square principle holds under ZFC + V=L, we can combine the result of Hamkins with Theorem 4.6 to get the following conclusion.

THEOREM 5.3. *Assume ZFC + V=L, and let X be a scattered Hausdorff space. Then X is Stäckel-compact if and only if it is countably compact. Hence it is relatively consistent with ZFC that all scattered countably compact Hausdorff spaces are Stäckel-compact.*

As mentioned earlier, while Theorem 5.3 is restricted to scattered spaces, Mrówka spaces (see [1]) provide examples of scattered spaces which distinguish different notions of compactness. So this limits the types of spaces in which a non-Stäckel-compact countably compact space may be found under ZFC, further raising the possibility that Theorem 5.3 may hold in ZFC (without assuming V=L). I thank the referee for this observation.

At the suggestion of the referee, we end this section by enclosing the relevant parts from [5] here for completeness.

DEFINITION (see [4, Definition 19]). The global square principle $\square$ is the assertion that there is an assignment $\nu \mapsto C_\nu$, for all singular ordinals ν, such that

- C_ν is a closed subset of ν, containing only singular ordinals;
- if ν has uncountable cofinality, then C_ν is unbounded in ν;
- the order type of C_ν is less than ν;
- if $\mu \in C_\nu$, then $C_\mu = C_\nu \cap \mu$.

THEOREM (Hamkins [5]). *Under the global square principle $\square$, there is a global partition of the class of singular ordinals into $A \sqcup B$ such that for every κ of uncountable cofinality, both $A \cap \kappa$ and $B \cap \kappa$ are stationary in κ.*

Proof (Hamkins, reproduced from [5]). Fix the $\square$ sequence C_ν. First, define A and B up to ω_1 to be any partition of the singular countable ordinals into stationary sets. Suppose now that A and B are defined up to ν, a singular limit ordinal. Consider C_ν, which has some order type $\eta < \nu$. If $\eta \in A$, then put $\nu \in A$, otherwise, put $\nu \in B$. Continue by transfinite recursion. Note that A and B partition the singular ordinals.

Suppose that κ has uncountable cofinality. If $\kappa = \omega_1$, then $A \cap \kappa$ and $B \cap \kappa$ are the stationary sets that we used to start the construction. More generally, if $\kappa > \omega_1$ but κ has cofinality ω_1, then κ is singular and so C_κ is a club of some type $\beta < \kappa$. Further, A and B when restricted to C_κ are copies of $A \cap \beta$ and $B \cap \beta$, which by induction are each stationary. So $A \cap \kappa$ and $B \cap \kappa$ are stationary. Finally, we have the case that κ has cofinality larger than ω_1. Fix any club $C \subset \kappa$. Thus, there is some singular $\eta \in C$ with uncountable cofinality. So $C_\eta \cap C$ is club in η and thus meets both A and B, and so C meets both A and B, as desired. ■

6. Stäckel-compactness and Novák spaces. I am greatly indebted to the referee for the results of this section, as they were obtained after the referee suggested that Novák spaces may resolve the question of productivity of Stäckel-compact spaces, a question that was left open in the original version of the article. This was indeed the case and led readily to a negative answer to the question (Corollary 6.6). Moreover, Novák's method produces examples of non-compact Stäckel-compact spaces without using the theory of stationary sets (Proposition 6.5); all our earlier such examples needed stationary sets.

Let $\beta(\mathbb{N})$ denote the Stone–Čech compactification of the discrete space $\mathbb{N}$ of positive integers. For any set E, let $|E|$ denote its cardinality.

Definition 6.1. We will call a subspace X of $\beta(\mathbb{N})$ a *basic Novák space* if $X = \bigcup_{\xi<\omega_1} X_\xi$ for some ω_1-sequence $\langle X_\xi\rangle_{\xi<\omega_1}$ such that for all $\xi < \omega_1$, $\aleph_0 \le |X_\xi| \le 2^{\aleph_0}$, X_ξ is disjoint from $\bigcup_{\mu<\xi} X_\mu$, and every infinite subset of $\bigcup_{\mu<\xi} X_\mu$ has a limit point in X_ξ.

Corollary 6.2 (Immediate from Definition 6.1). *Let $X = \bigcup_{\xi<\omega_1} X_\xi$ be a basic Novák space with $\langle X_\xi\rangle_{\xi<\omega_1}$ as in Definition* 6.1.

(a) *For any infinite $S \subseteq X$ there is $\xi < \omega_1$ such that S has limit points in X_ν for all $\nu \ge \xi$.*
(b) *If $I \subseteq \omega_1$ is uncountable, then $\bigcup_{\xi\in I} X_\xi$ is also a basic Novák space.*

Proposition 6.3 (Novák). *If $A \subseteq \beta(\mathbb{N})$ and $\aleph_0 \le |A| \le 2^{\aleph_0}$, there is a basic Novák space $X = \bigcup_{\xi<\omega_1} X_\xi$ with $X_0 = A$ and $\langle X_\xi\rangle_{\xi<\omega_1}$ as in Definition* 6.1. *There are basic Novák spaces Y, Z such that $Y \times Z$ is not countably compact.*

Proof (outline, from Novák's construction [3]). To get X, take $X_0 = A$ and define via transfinite recursion the sets X_ξ, which can be chosen to meet the conditions of Definition 6.1 because any infinite $S \subseteq \beta(\mathbb{N})$ has $2^{2^{\aleph_0}}$ limit points, and if $|S| \le 2^{\aleph_0}$ then $|\{E \subseteq S : |E| \le \aleph_0\}| \le 2^{\aleph_0}$. Now take such a basic Novák space X with $X_0 = \mathbb{N}$, fix uncountable disjoint $I, J \subseteq \omega_1 \setminus \{0\}$, and let $Y := \bigcup_{\xi\in I\cup\{0\}} X_\xi$ and $Z := \bigcup_{\xi\in J\cup\{0\}} X_\xi$. This gives basic Novák spaces Y and Z with $Y \cap Z = \mathbb{N}$. Then $Y \times Z$ is not countably compact, since it has an infinite discrete closed subset $\{(n, n) : n \in \mathbb{N}\}$. ■

Lemma 6.4. *Let X be a Hausdorff topological space containing two disjoint sets A and B such that every infinite subset of X has a limit point in A and also a limit point in B. Then X is Stäckel-compact.*

Proof. Fix a linear order on X in which the set A wholly precedes the set $X \setminus A$, the set A is well-ordered, and the set $X \setminus A$ is reverse well-ordered. ∎

PROPOSITION 6.5. *Every basic Novák space X is Stäckel-compact.*

Proof. Let X be a basic Novák space and express it as $X = \bigcup_{\xi<\omega_1} X_\xi$ with $\langle X_\xi \rangle_{\xi<\omega_1}$ as in Definition 6.1. Fix uncountable disjoint $I, J \subseteq \omega_1$, and let $A := \bigcup_{\xi\in I} X_\xi$ and $B := \bigcup_{\xi\in J} X_\xi$. The sets A, B are disjoint, and every infinite subset of X has limit points in A and also in B (Corollary 6.2 (a)). So X is Stäckel-compact by Lemma 6.4. ∎

COROLLARY 6.6. *There are Stäckel-compact spaces Y, Z such that $Y \times Z$ is not countably compact, so Stäckel-compactness is not a productive property.*

REMARK. A basic Novák space $X = \bigcup_{\xi<\omega_1} X_\xi$ is a subspace of $\beta(\mathbb{N})$ of size $2^{\aleph_0}$, and gives an example of a Stäckel-compact space that is neither compact nor sequentially compact (unlike ω_1).

7. Open questions, credits, and history

Open questions. We list some problems unanswered in this article.

(1) If X is a countably compact Hausdorff space, is X Stäckel-compact? *This is the most significant question we have left unsettled.*
(2) Can we answer (1) if we also assume that X is a Tikhonov space?
(3) Is every Stäckel-compact space (T_2 by definition) regular?
(4) Is the continuous image of a Stäckel-compact space in a Hausdorff space necessarily Stäckel-compact?

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