Linear Algebra Practice Problems

These problems cover Chapters 4, 5, 6, and 7 of *Elementary Linear Algebra,* 6th ed, by Ron Larson and David Falvo (ISBN-13 = 978-0-618-78376-2, ISBN-10 = 0-618-78376-8). Direct questions from Chapters 1–3 do not appear here, but the topics of Chapters 1–3 are very important because the techniques covered there (Gaussian and Gauss-Jordan elimination, matrix algebra, determinants) are essential for the later chapters.

1. True or False?

(a) A linear transformations is completely determined by its values on a basis for the domain.

(b) The kernel of a linear transformation is a subspace of the domain.

(c) The range of a linear transformation is a subspace of the co-domain.

(d) The rank of a linear transformation equals the dimension of its kernel.

(e) The nullity of a linear transformation equals the dimension of its range.

(f) A linear transformation T is one-to-one if and only if $\ker(T) = \{0\}$.

(g) If $T\colon V \to \mathbf{R}^5$ is a linear transformation then T is onto if and only if $\operatorname{rank}(T) = 5$.

(h) If a linear transformation $T\colon \mathbf{R}^n \to \mathbf{R}^n$ is one-to-one, then it is onto and hence an isomorphism.

(i) If a linear transformation $T\colon \mathbf{R}^n \to \mathbf{R}^n$ is onto, then it is one-to-one and hence an isomorphism.

2. If $T\colon \mathbf{R}^2 \to \mathbf{R}$ is a linear transformation from the plane to the real numbers and if $T(1,1) = 1$ and $T(-1,0) = 2$, then $T(3,5)$ equals:

(A) -6 (B) -5 (C) 0 (D) 8 (E) 9

3. Let $V = M_{2,3}$ be the vector space of all 2×3 matrices, and let $W = M_{4,1}$ be the vector space of all 4×1 column vectors. If T is a linear transformation from V **onto** W, what is the dimension of $\{\mathbf{v} \in V : T(\mathbf{v}) = 0\}$?

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

4. Let $\begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 1 \end{bmatrix}$ be the standard matrix of T. If $\mathbf{e}_2 = (0,1,0)$ and $\mathbf{v} = (1,1,1)$, find

(a) $T(\mathbf{e}_2)$

(b) $T(\mathbf{v})$

(c) $T(x,y,z)$.

5. For each of the following linear transformations find its standard matrix.

(a) $T\colon \mathbf{R}^2 \to \mathbf{R}^3$, defined by $T(x,y) = (3y,\ 2x,\ x - 4y)$.

(b) $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ is the reflection across x-axis.

(c) $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ is the orthogonal projection onto the line $y = 2x$.

6. Consider the space $C[-1,1]$ of all continuous functions on $[-1,1]$ as an inner product space with the inner product defined by
$$\langle f, g\rangle := \int_{-1}^{1} f(x)g(x)\,dx.$$
Let ϕ and ψ be be given by:
$$\phi(x) := 1 \qquad (-1 \le x \le 1), \qquad\qquad \text{and} \qquad\qquad \psi(x) := 3x - 1 \qquad (-1 \le x \le 1).$$
 (a) Find the angle between ϕ and ψ.
 (b) Find $\operatorname{proj}_{\psi}\phi$ (the orthogonal projection of ϕ onto ψ), and $\operatorname{proj}_{\phi}\psi$ (projection of ψ onto ϕ).
 (c) Find an orthonormal basis for $\operatorname{span}(\{\phi, \psi\})$.

7. Let T be the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$, where
$$A = \begin{bmatrix} 1 & 2 & 2 & 3 & 1 \\ 2 & 4 & 5 & 5 & 4 \\ 3 & 6 & 7 & 8 & 5 \end{bmatrix}.$$
 (a) Find a basis for $\ker(T)$.
 (b) Find $\text{nullity}(T)$
 (c) Find a basis for $\text{range}(T)$.
 (d) Find $\text{rank}(T)$

8. Let $M_{2,3}$ be the vector space of all 2×3 matrices, let $T\colon M_{2,3} \to \mathbf{R}^3$ be a linear transformation with domain $M_{2,3}$. (Thus for each 2×3 matrix A, $T(A)$ is a vector in $\mathbf{R}^3$.) Suppose that the range of T is the xz-plane $\{(x,y,z)\colon y = 0\}$. Find the rank of T and the dimension of $\ker(T)$ (the nullity of T), giving detailed reasons for your answers.

9. Notation: If B is a given basis of a finite dimensional vector space V, then for every vector $\mathbf{x}$ in V, we write $[\mathbf{x}]_B$ to denote the coordinates of $\mathbf{x}$ with respect to the basis B.

   Now consider two bases B and B' for $\mathbf{R}^2$, where $B = \{(2,1), (3,2)\}$, and B' is the standard basis.
 (a) Suppose that the vector $\mathbf{x} \in \mathbf{R}^2$ has coordinates $(-2,1)$ with respect to the (non-standard) basis B, i.e. $[\mathbf{x}]_B = (-2,1)$. Find $[\mathbf{x}]_{B'}$, the coordinates of $\mathbf{x}$ with respect to the standard basis.
 (b) Find the transition matrix from B to B', i.e. find a matrix P such that $P[\mathbf{x}]_B = [\mathbf{x}]_{B'}$ for every vector $\mathbf{x}$ in $\mathbf{R}^2$.
 (c) Find the transition matrix from B' to B, i.e. find a matrix Q such that $Q[\mathbf{x}]_{B'} = [\mathbf{x}]_B$ for every vector $\mathbf{x}$ in $\mathbf{R}^2$.
 (d) Suppose that the vector $\mathbf{x} \in \mathbf{R}^2$ has coordinates $(-2,1)$ with respect to the standard basis B', i.e. $[\mathbf{x}]_{B'} = (-2,1)$. Find $[\mathbf{x}]_B$, the coordinates of $\mathbf{x}$ with respect to the (nonstandard) basis B.
 (e) If $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ has standard matrix $A = \begin{bmatrix} 3 & 3 \\ 5 & 4 \end{bmatrix}$, find the matrix of T relative to the basis B.

10. For each matrix A, diagonalize A if possible, following the steps listed below.

(a) $A = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (c) $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ (d) $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{bmatrix}$.

(a) Find all eigenvalues and eigenvectors, and bases for each eigenspace.

(b) List algebraic and geometric multiplicities of each eigenvalue.

(c) Determine if A can be diagonalized.

(d) If A can be diagonalized, find a matrix P such that $P^{-1}AP$ is diagonal.

11. For each matrix A given, orthogonally diagonalize A following the steps listed below.

(a) $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$; (b) $A = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$; (c) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

(a) Find the characteristic equation and all eigenvalues of A, listing the multiplicity of each eigenvalue.

(b) For each eigenvalue λ of A, find all the eigenvectors (the eigenspace) corresponding to it.

(c) Find a basis consisting of an **orthogonal set of eigenvectors** of A.

(d) Using the results above, orthogonally diagonalize A by finding a suitable orthogonal matrix P such that $P^{-1}AP$ is diagonal.

12. Consider the vectors $\mathbf{u} = (1, -1, -1, 0)$ and $\mathbf{v} = (1, -1, 0, -1)$ in $\mathbf{R}^4$ with the standard inner product (i.e. the dot product). Let W be the subspace (of $\mathbf{R}^4$) spanned by $\mathbf{u}$ and $\mathbf{v}$, and let $W^\perp$ be the subspace of all vectors orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. Find an orthonormal basis $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$, $\mathbf{w}_4$ of $\mathbf{R}^4$ such that $\mathbf{w}_1$, $\mathbf{w}_2$ is an orthonormal basis for W and $\mathbf{w}_3$, $\mathbf{w}_4$ is an orthonormal basis for $W^\perp$.

13. In $\mathbf{R}^3$, let $S = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right)$.

(a) Find the orthogonal complement $S^\perp$ of the set S (all vectors orthogonal to all vectors of S).

(b) Find a linear transformation whose kernel is S and whose range is $S^\perp$.

(c) Find a linear transformation whose kernel is $S^\perp$ and whose range is S.

14. Let $T\colon \mathbf{R}^3 \to \mathbf{R}^3$ be the transformation on $\mathbf{R}^3$ which reflects every vector across the plane $x+y+z = 0$.

(a) List all eigenvalues of T.

(b) Describe all the eigenvectors of T.

(c) For each eigenvalue, find its algebraic and geometric multiplicity.

(d) Write down the standard 3×3 matrix of T.

Multiple Choice Problems

15. The dimension of the subspace spanned by the real vectors

$$\begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\2\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\0\\0\\3\end{bmatrix}, \begin{bmatrix}1\\-2\\0\\8\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \quad \text{is}$$

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

16. The rank of the matrix

$$\begin{bmatrix}1 & 2 & 3 & 4 & 5\\ 6 & 7 & 8 & 9 & 10\\ 11 & 12 & 13 & 14 & 15\\ 16 & 17 & 18 & 19 & 20\\ 21 & 22 & 23 & 24 & 25\end{bmatrix} \quad \text{is}$$

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

17. If M is the matrix $\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{bmatrix}$, then M^{100} is:

(A) $\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{bmatrix}$ (B) $\begin{bmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}$ (C) $\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ (D) $\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$ (E) none of the above

18. Let A and B be subspace of a vector space V. Which of the following must be subspaces of V?

I. $A + B := \{\mathbf{x} + \mathbf{y} : \mathbf{x} \in A \text{ and } \mathbf{y} \in B\}$
II. $A \cup B$
III. $A \cap B$
IV. $\{\mathbf{x} \in V : x \notin A\}$

(A) I and II only
(B) I and III only
(C) III and IV only
(D) I, II, and III only
(E) I, II, III, and IV

19. If $A = \begin{bmatrix}1 & 2\\ 0 & -1\end{bmatrix}$ then the set of all vectors $\mathbf{x}$ for which $A\mathbf{x} = \mathbf{x}$ is

(A) $\{[a, b]^T : a = 0 \text{ and } b \text{ is arbitrary}\}$
(B) $\{[a, b]^T : a \text{ is arbitrary and } b = 0\}$
(C) $\{[a, b]^T : a = -b \text{ and } b \text{ is arbitrary}\}$
(D) $\{[0, 0]^T\}$
(E) The empty set

20. Let P_3 be the vector space of all real polynomials that are of degree at most 3. Let W be the subspace of all polynomials $p(x)$ in P_3 such that $p(0) = p(1) = p(-1) = 0$. Then $\dim(V) + \dim(W)$ is

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

21. Let T be the transformation of the xy-plane that reflects each vector through the x-axis then doubles the vector's length. If A is the (standard) matrix of T, then $A =$

(A) $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$

(B) $\begin{bmatrix} \frac{\sqrt{2}}{2} & 1 \\ 1 & -\frac{\sqrt{2}}{2} \end{bmatrix}$

(C) $\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

(D) $\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$

(E) $\begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}$

22. Let V be the vector space of real valued functions defined on the real numbers and having derivatives of all orders. If D is the mapping from V into V that maps every function in V to its derivative, what are all the eigenvectors of D?

(A) All non-zero functions in V

(B) All non-zero constant functions in V

(C) All non-zero functions of the form $ke^{\lambda x}$, where k and λ are real numbers

(D) All non-zero functions of the form $\sum_{i=0}^{k} c_i x^i$, where $k > 0$ and the c_i's are real numbers

(E) There are no eigenvectors of D

23. For what value (or values) of m is the vector $(1, 2, m, 5)$ a linear combination of the vectors $(0, 1, 1, 1)$, $(0, 0, 0, 1)$, and $(1, 1, 2, 0)$?

(A) For no value of m

(B) -1 only

(C) 1 only

(D) 3 only

(E) For infinitely many values of m

24. Which of the following sets of vectors is a basis for the subspace of Euclidean 4-space consisting of all vectors that are orthogonal to both $(0, 1, 1, 1)$ and $(1, 1, 1, 0)$?

(A) $\{(0, -1, 1, 0)\}$

(B) $\{(1, 0, 0, 0), (0, 0, 0, 1)\}$

(C) $\{(-2, 1, 1, -2), (0, 1, -1, 0)\}$

(D) $\{(1, -1, 0, 1), (-1, 1, 0, -1), (0, 1, -1, 0)\}$

(E) $\{(0, 0, 0, 0), (-1, 1, 0, -1), (0, 1, -1, 0)\}$

25. Suppose that B is a basis for a vector space V of dimension greater than 1. Which of the following statements could be true?

    (A) The zero vector of V is an element of B.

    (B) B has a proper subset that spans V.

    (C) B is a proper subset of a linearly independent subset of V.

    (D) There is a basis for V that contains no vector of B.

    (E) One of the vectors in B is a linear combination of the other vectors in B.

26. If A is a 3×3 matrix such that $A \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $A \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, then the product $A \begin{bmatrix} 6 \\ 7 \\ 8 \end{bmatrix}$ is

    (A) $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ (B) $\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ (C) $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ (D) $\begin{bmatrix} 9 \\ 10 \\ 11 \end{bmatrix}$ (E) Not uniquely determined

Proofs

Proofs must be correct, clear, complete, and precise.

27. Prove the following.

    (a) Let $T \colon V \to V$ be a linear transformation and suppose that the set of vectors $\mathbf{v}$ such that $T(\mathbf{v}) = \mathbf{v}$ is a spanning set. Then T must be the identity mapping, i.e. $T(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in V$.

    (b) Any two eigenvectors (non-zero) having *distinct* eigenvalues must be linearly independent.

    (c) Any number of eigenvectors (non-zero) with *all distinct* eigenvalues must be linearly independent.

    (d) Let $T \colon V \to W$ be a linear transformation. Then:

    - $\text{Ker}(T)$ is a subspace of V and $\text{Ran}(T)$ is a subspace of W.
    - T is one-to-one if and only if $\text{Ker}(T) = \{\mathbf{0}\}$ (i.e. if and only if $\dim(\text{Ker}(T)) = 0$).
    - If $\dim(W) = n$ then T is onto if and only if $\text{Ran}(T)$ spans W if and only if $\dim(\text{Ran}(T)) = n$.
    - If T is one-to-one, then T maps linearly independent vectors to linearly independent vectors.

    (e) Let $T \colon V \to V$ be a linear transformation. Then:

    - If λ is an eigenvalue of T, then the eigenspace $E_\lambda := \{\mathbf{v} \mid T\mathbf{v} = \lambda\mathbf{v}\}$ is a subspace of V.
    - T is one-to-one if and only if 0 is not an eigenvalue of T.

28. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in a vector space V. Prove that:

    (a) $\mathbf{u}$ and $\mathbf{v}$ are linearly independent if and only if $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are linearly independent.

    (b) The three vectors $\mathbf{u} - \mathbf{v}$, $\mathbf{v} - \mathbf{w}$, and $\mathbf{w} - \mathbf{u}$ are linearly dependent.

    (c) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ form a basis for V, then the vectors $\mathbf{u}$, $\mathbf{u} + \mathbf{v}$, and $\mathbf{u} + \mathbf{v} + \mathbf{w}$ form another basis for V.

29. Show that if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are non-zero vectors in an inner product space which are pairwise orthogonal (i.e. $\mathbf{v}_i \perp \mathbf{v}_j$ for $1 \leq i \neq j \leq n$), then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent.

30. Let V be the vector space of all real polynomials $p(x)$. Let transformations T, S be defined on V by $T\colon p(x) \to xp(x)$ and $S\colon p(x) \to p'(x) = \frac{d}{dx}p(x)$. Interpret $(ST)(p(x))$ as $S(T(p(x)))$, and similarly for TS.

   (a) Prove that S and T are linear transformations.

   (b) Find the kernel and the range for each of S and T.

   (c) Prove that ST is an isomorphism of V onto V (that is, ST is both one-to-one and onto).

The following equivalences are very useful. It may be instructive to practice the proofs for all the equivalences below.

If $T\colon R^n \to R^n$ is a linear transformation with matrix A, then the following conditions are all equivalent:

1. T is one-to-one
2. The only solution of the homogeneous system $A\mathbf{x} = \mathbf{0}$ is the trivial solution $\mathbf{x} = \mathbf{0}$
3. T has zero nullity, i.e. $\dim(\mathrm{Ker}(T)) = 0$
4. T has full rank, i.e. $\dim(\mathrm{Range}(T)) = n$
5. T is onto
6. The system $A\mathbf{x} = \mathbf{b}$ has a solution for every vector $\mathbf{b}$
7. T is both one-to-one and onto
8. The system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every vector $\mathbf{b}$
9. T is an isomorphism of V onto V
10. T maps some basis to a basis
11. T maps every basis to a basis.
12. A is non-singular (or equivalently, A is invertible, i.e. A^{-1} exists)
13. The columns of A are linearly independent
14. The rows of A are linearly independent
15. The columns of A span $\mathbf{R}^n$
16. The rows of A span $\mathbf{R}^n$
17. $\det(A) \neq 0$