ERRATA LIST FOR SET THEORY

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The following is the list of errata for the book *Set Theory: With an Introduction to Real Point Sets,* by Abhijit Dasgupta, Birkhäuser/Springer, 2014 (ISBN 978-1-4614-8853-8, hardcover).

- **p. ix**: Add the following sentence at the end of the top paragraph: "In particular, *the postscript chapters fall outside the main development followed in the book.*
- **p. 9**: Line before Problem 16: "xRu and uRy" should be "xRu and uSy".
- **pp. 42–43: Error in the proof of Theorem 146:**

  The author wishes to thank Dr. Christoph Lamm for noticing this error. The rigorous definition of addition given in Section 2.10 (using primitive recursion) is based ultimately on Theorem 146 (Basic Principle of Recursive Definition). The proof of Theorem 146 assumes that the ordering relation $<$ on $\mathbf{N}$ has already been defined. But the definition of ordering on $\mathbf{N}$ given earlier (Definition 61) had presupposed addition itself, resulting in *circular reasoning.*

  The proof of Theorem 146 can be corrected by defining the set I_n of the first n natural numbers *without presupposing the ordering relation* $<$. A complete replacement for pages 42–43 with this correction to the proof of Theorem 146 appears at the end of this errata list.
- **p. 49**: After the figure, remove the extra space (indentation) preceding the phrase "if both legs ...".
- **p. 60**: Top line of text: Two union symbols after $\{0\}$ are missing: In each case, "$\{0\}\{$" should become "$\{0\} \cup \{$".
- **p. 83**: Item 2 in "**Informal discussion**": "...called bounded if we have $-a < x < a$ for some real number a" should be changed to: "...called bounded if there is a real number a such that $-a < x < a$ for all $x \in E$."
- **p. 92**: Paragraph after Problem 290, second line: "congruence and similarity mappings" should be "congruence and similarity of geometric figures".
- **p. 97**: Problem 313: $\frac{1}{2}(m+n)(m+n-1)+m$ should be $\frac{1}{2}(m+n-2)(m+n-1)+m$, and similarly, the displayed formula should become

$$\langle m, n\rangle \mapsto \frac{(m+n-2)(m+n-1)}{2} + m$$

- **p. 111**: After Theorem 350, add the following sentence: "By Theorem 350, the cardinal comparison symbols $\leqslant$ and $\leq$ become equivalent, and so we can (and henceforth will) use them interchangeably."

- **p. 124**: Problem 418, last clause in the casewise definition for $\mathbf{\Phi}$: Instead of x it should be $\mathbf{h}(x)$.
- **p. 166**: The displayed sentence "*The Suslin Problem. Is a CCC ...*" is all in italics, but the first three words should be upright bold, as in: "**The Suslin Problem**. *Is a CCC ...*".
- **p. 207**: In Theorem 714 part 1, change "*but* $X^{(\alpha)} \subsetneq X^{(\mu)}$ *for* $\alpha < \mu$" to "*but* $X^{(\alpha)} \supsetneq X^{(\mu)}$ *for all* $\alpha < \mu$".
- **p. 249**: *Proof (of Theorem 877, Outline)*: In 5th line of the proof, "with $(c, d) \cap E_\alpha \neq \emptyset$ ..." should be "with $(c, d) \cap E_\alpha = \emptyset$ ...".
- **p. 315**: Section 17.2, third paragraph, line 3: "homeomorphism of F_1 onto itself which interchanges ..." should be "homeomorphism of $\mathbf{K}_1$ onto itself which interchanges ...".
- **p. 323**: In Problem 1126, Hint: "$B_E := \cup_{n \in E} A_n \setminus \cup_{n \notin E} A_n$ ..." should be "$B_E := \cap_{n \in E} A_n \setminus \cup_{n \notin E} A_n$ ..."
- **p. 346**: Beginning of 4th line of second paragraph of the proof of Theorem 1180: "with $\mu(G \setminus A) < \mu(A)$" should be "with $\mu(G \setminus B) < \mu(A)$".
- **p. 347**: Definition 1185: $p \neq q$ should be $x \neq y$.
- **p. 420**: End of proof of Proposition 1335: "$\geq m^*(A) + m^*(B)$" should be "$\geq m^*(A) + m^*(B) - \epsilon$".
- **p. 431**:
  - Change index entry "absolutism, 67" to "absolutism, 67, 70"
  - Under the index entry for "axiom of":
    * Change subentry "choice, 13, 77, 90–94, 208–210" to "choice, 13, 90–94, 208–210, 224–225"
    * Change subentry "dependent choice (DC), 77, 101" to "dependent choice (DC), 101"
- **p. 432**:
  - Under the index entry for "axiom of":
    * Change subentry "foundation, 393–395, 421" to "foundation, 393–394, 421"
    * Change subentry "replacement, 376–377, 421" to "replacement, 376–377, 409, 421"
  - Change index entry "Bernstein sets, 298–299, 335, 345–346, 408" to "Bernstein sets, 298–299, 335, 346, 402, 408"

2.10 Recursive Definitions*

Recall that we had "defined" addition of natural numbers by the following *recursion equations*:

$$m+1 := S(m), \qquad \text{and} \qquad m+S(n) := S(m+n).$$

But this is not an explicit definition! We took it for granted (as was done in the work of Peano) that a two-place function + (the mapping $(m,n) \mapsto m+n$) satisfying the above equations exists, without giving any rigorous justification for its existence. Similarly, multiplication of natural numbers was "defined" by recursion equations without proper justification.

Dedekind introduced a general method, known as *primitive recursion,* which provides such justification. It assures the *existence and uniqueness* of functions which are defined implicitly using recursion equations having forms similar to the ones for addition and multiplication.

We will formulate and prove a general version of Dedekind's principle of recursive definition, from which the existence and uniqueness for the sum and product functions can be immediately derived.

Principles of Recursive Definition

The following *Basic Principle of Recursive Definition* is perhaps the simplest yet very useful result for defining functions recursively.

Theorem 146 (Basic Principle of Recursive Definition). *If Y is a set, $a \in Y$, and $h\colon Y \to Y$, then there is a unique $f\colon \mathbf{N} \to Y$ such that*

$$f(1) = a, \qquad \textit{and} \qquad f(S(n)) = h(f(n)) \ \textit{for all } n \in \mathbf{N}.$$

Informally, this says that given $a \in Y$ and $h\colon Y \to Y$, we can form the infinite sequence $\langle a, h(a), h(h(a)), \ldots \rangle$.

Proof. The uniqueness of the function f can be established by an easy and routine induction, so let us prove existence.

A subset I of $\mathbf{N}$ will be called an *initial set* if for all $k \in \mathbf{N}$, if $S(k) \in I$ then $k \in I$. By routine induction, we can establish the following:

- $\{1\}$ is an initial set, and every non-empty initial set contains 1 as a member.
- If I is an initial set with $k \in I$ then $I \cup \{S(k)\}$ is also an initial set.
- For each $n \in \mathbf{N}$, there is a unique initial set I such that $n \in I$ but $S(n) \notin I$.

Let I_n denote the unique initial set containing n but not $S(n)$. It follows that $I_1 = \{1\}$, and $I_{S(n)} = I_n \cup \{S(n)\}$ for all $n \in \mathbf{N}$. (Informally, $I_n = \{1,2,\ldots,n\}$, the set of first n natural numbers.) The proof will use *functions $u\colon I_n \to Y$ having domain I_n*.

Let us say that a function *u is partially h-recursive with domain I_n* if $u\colon I_n \to Y$, $u(1) = a$, and $u(S(k)) = h(u(k))$ for all k such that $S(k) \in I_n$.

We first prove by induction that for every $n \in \mathbf{N}$ there is a unique partially h-recursive u with domain I_n.

Basis step ($n = 1$): Let $v\colon \{1\} \to Y$ be defined by setting $v(1) = a$. Then v is partially h-recursive with domain I_1. Moreover, if $u,u'\colon I_1 \to Y$ are partially h-recursive functions with domain I_1, then $u(1) = a = u'(1)$, so $u = u'$ since 1 is the only element in their domain $I_1 = \{1\}$. So there is a unique partially h-recursive v with domain I_1, establishing the basis step.

Induction step: Suppose that $n \in \mathbf{N}$ is such that there is a unique partially h-recursive v with domain I_n (induction hypothesis). We fix this v for the rest of this step, and define $w\colon I_{S(n)} \to Y$ by setting $w(k) := v(k)$ for $k \in I_n$ and $w(k) := h(v(n))$ if $k = S(n)$. Then w is easily seen to be partially h-recursive with domain $I_{S(n)}$. Moreover, if $u,u'\colon I_{S(n)} \to Y$ are partially h-recursive with domain $I_{S(n)}$, then the restrictions $u\restriction_{I_n}$ and $u'\restriction_{I_n}$ are partially h-recursive with domain I_n, so they must be identical by induction hypothesis, i.e. $u(k) = u'(k)$ for $k \in I_n$. In particular, $u(n) = u'(n)$, so $u(S(n)) = h(u(n)) = h(u'(n)) = u'(S(n))$, which gives $u = u'$. Hence there is a unique partially h-recursive w with domain $I_{S(n)}$, which finishes the induction step.

Thus for each n there is a unique partially h-recursive function with domain I_n; let us denote this function by u_n.

Now define $f\colon \mathbf{N} \to Y$ by setting:

$$f(n) := u_n(n).$$

First, $f(1) = a$ since $u_1(1) = a$. Next, the restriction of $u_{S(n)}$ to I_n equals u_n (by uniqueness, since the restriction is partially h-recursive), so $u_{S(n)}(n) = u_n(n)$. Hence $f(S(n)) = u_{S(n)}(S(n)) = h(u_{S(n)}(n)) = h(u_n(n)) = h(f(n))$. Thus f satisfies the recursion equations of the theorem. ◻

To handle functions of multiple variables, the following theorem is used.

Theorem 147 (General Principle of Recursive Definition). *For any $g\colon X \to Y$ and $h\colon X \times \mathbf{N} \times Y \to Y$, there is a unique function $f\colon X \times \mathbf{N} \to Y$ such that for all $x \in X$ and $n \in \mathbf{N}$:*

$$f(x,1) = g(x) \quad \text{and} \quad f(x,S(n)) = h(x,n,f(x,n)).$$

Here f is being defined by recursion on the second variable n, that is, n is the *variable of recursion* ranging over $\mathbf{N}$, while x is a *parameter* ranging over the set X. This is the most general form of recursive definition, where both the parameters (in X) and the values (in Y) come from arbitrary sets.

Proof. The proof is essentially the same as that of Theorem 146, since the additional parameter does not play any significant role in the recursion. The details are left as an exercise for the reader. ◻

Theorem 148 (Course of Values Recursion). *Let Y be a non-empty set and Y^* denote the set of all finite sequences (strings) of elements from Y. Given any $G\colon Y^* \to Y$ there is a unique $f\colon \mathbf{N} \to Y$ such that*