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# LIMIT POINTS OF DISCRETE SETS IN METRIC SPACES

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It is easy to see that in a metric space, the set of limit points of a discrete subset is a nowhere dense closed set. It is an interesting fact that the converse is also true in any metric space. Since a set is nowhere dense closed if and only if it is the boundary of some open set, the result can be stated as:

**Problem** (Exercise 4G in Willard's *General Topology*)**.** In a metric space, the boundary of an open set is the set of limit points of some discrete subset.

It is not hard to derive this using the paracompactness property of metric spaces. But in Willard's book, the problem is assigned in Section 4, long before metrization and paracompactness notions are even mentioned. In this note we present a solution of the problem which does not use paracompactness of metric spaces.

In fact, we prove the following slightly stronger version of the problem.

**Proposition.** In a metric space, the boundary of an open set $G$ is the set of limit points of some discrete subset of $G$.

*Proof.* Let $d$ be the metric, $B[p, r]$ denote the open ball $\{x : d(x, p) < r\}$, and for any set $S$, let $d(p, S) = \inf_{s\in S} d(p, s)$, and let $S'$ denote the set of limit points of $S$.

Suppose that $A$ is the boundary of an open set $G$. We assume that $A \neq \emptyset$, as otherwise the result is trivial. Then $G \neq \emptyset$, and $A = \overline{G} \smallsetminus G = G' \smallsetminus G$. Let

$$G_n = \left\{ x \in G : d(x, A) < \frac{1}{n} \right\} \quad (n = 1, 2, 3, \ldots).$$

Then $G_n$ is a non-empty open subset of $G$ with $A \subseteq G'_n$, $A \cap G_n = \emptyset$, and $G_{n+1} \subseteq G_n$ (for $n = 1, 2, 3, \ldots$). Moreover, it is easily verified that $A = \bigcap_{n=1}^{\infty} G'_n$. For any $\epsilon > 0$, call a set $S$ to be *$\epsilon$-sparse* iff for all distinct $x, y \in S$, $d(x, y) \geq \epsilon$. And call $S$ *sparse* iff $S$ is $\epsilon$-sparse for some $\epsilon > 0$.

Some properties of sparse sets are:

(a) If $S$ is sparse, then $S' = \emptyset$, so $S$ is discrete.
(b) A finite union of sparse sets is discrete. This follows from the fact that $(S \cup T)' = S' \cup T'$.
(c) Union of two sparse sets may fail to be sparse! For example in $\mathbb{R}$, each of the sets $\mathbb{Z}$ and $\sqrt{2}\mathbb{Z} = \{n\sqrt{2} : n \in \mathbb{Z}\}$ is sparse but $\mathbb{Z} \cup \sqrt{2}\mathbb{Z}$ is not sparse.
(d) For any *fixed* $\epsilon > 0$, the class of $\epsilon$-sparse sets is closed under chain unions, that is the union of any chain (linearly ordered under inclusion) of $\epsilon$-sparse sets is an $\epsilon$-sparse set.

Using the last property and Zorn's lemma, choose a maximal $\frac{1}{n}$-sparse subset $S_n$ of the non-empty open set $G_n$ ($n = 1, 2, 3, \ldots$), and let $D = \bigcup_{n=1}^{\infty} S_n$. We show that $D$ is the desired set, i. e., $D$ is discrete with $D' = A$.

For all $n$,

$$\begin{aligned}
D' &= \left[S_1 \cup S_2 \cup \ldots \cup S_n \cup \left(\bigcup_{j=n+1}^{\infty} S_j\right)\right]' \\
&= S_1' \cup S_2' \cup \ldots \cup S_n' \cup \left(\bigcup_{j=n+1}^{\infty} S_j\right)' \\
&= \left(\bigcup_{j=n+1}^{\infty} S_j\right)' && \text{(since each } S_i \text{ is sparse, } S_i' = \emptyset) \\
&\subseteq \left(\bigcup_{j=n+1}^{\infty} G_j\right)' \\
&= G_{n+1}' && \text{(since } G_{n+1} \supseteq G_{n+2} \supseteq \ldots) \\
&\subseteq G_n', && \text{(since } G_{n+1} \subseteq G_n)
\end{aligned}$$

so $D' \subseteq G_n'$ for all $n$, so $D' \subseteq A$. But $D \subseteq G$ and $A \cap G = \emptyset$, so $D' \cap D = \emptyset$, hence $D$ is discrete.

It remains to show that $A \subseteq D'$. Since $A \cap D = \emptyset$, this is same as showing $A \subseteq \overline{D}$. Let $x \in A$ and $\epsilon > 0$ be given. We will prove:

$$B[x, \epsilon] \cap D \neq \emptyset. \tag{1}$$

Choose $n$ such that $\frac{1}{n} < \frac{\epsilon}{2}$. Since $A \subseteq \overline{G}$, $B[x, \frac{1}{n}] \cap G \neq \emptyset$. Fix $y \in B[x, \frac{1}{n}] \cap G$. Then $d(x, y) < \frac{1}{n}$, so $y \in G_n$.

Since $S_n$ is a maximal $\frac{1}{n}$-sparse subset of $G_n$, there exists $z \in S_n$ with $d(y, z) < \frac{1}{n}$ (as otherwise $S_n \cup \{y\}$ would be a $\frac{1}{n}$-sparse subset of $G_n$ strictly containing $S_n$). Then $d(x, z) \leq d(x, y) + d(y, z) < \frac{1}{n} + \frac{1}{n} < \epsilon$, so $z \in B[x, \epsilon] \cap D$, proving (1). □

**Note.** If we use the fact that every metrizable space is paracompact, then the following simpler method can be used.

*Proof.* Let $A$ be the boundary of an open set $G$. Put $U_x = G \cap B[x, d(x, A)/2]$ for $x \in G$, so that $\mathcal{U} = \{U_x : x \in G\}$ is an open cover of $G$. By paracompactness of $G$, $\mathcal{U}$ will have a refinement $\mathcal{V}$ such that $\mathcal{V}$ consists of non-empty open sets, $\mathcal{V}$ covers $G$, and $\mathcal{V}$ is locally-finite over $G$. For each $V \in \mathcal{V}$, pick a point $p_V \in V$, and let $D = \{p_V : V \in \mathcal{V}\}$. Then $D$ is a discrete subset of $G$ with $D' = A$. □

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