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COUNTABLE METRIC SPACES WITHOUT ISOLATED POINTS

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Theorem (Sierpinski, 1914–1915, 1920)**.** Any countable metrizable space without isolated points is homeomorphic to $\mathbb{Q}$, the rationals with the order topology (same as $\mathbb{Q}$ as a subspace of $\mathbb{R}$ with usual topology, or as $\mathbb{Q}$ with the metric topology).

The theorem is remarkable, and gives some apparently counter-intuitive examples of spaces homeomorphic to the usual $\mathbb{Q}$. Consider the "Sorgenfrey topology on $\mathbb{Q}$," which has the collection $\{(p, q] : p, q \in \mathbb{Q}\}$ as a base for its topology. This topology on $\mathbb{Q}$ is strictly finer than, and yet homeomorphic to, the usual topology of $\mathbb{Q}$. Another example is $\mathbb{Q} \times \mathbb{Q}$ as a subspace of the Euclidean plane.

In this article, we present three proofs of Sierpinski's theorem.

1. Order-theoretic proof

This proof is fairly elementary in the sense that no "big guns" are used (such as Brouwer's characterization of the Cantor space or the Alexandrov-Urysohn characterization of the irrationals), and no new back-and-forth method is used, but the main tool is:

Theorem (Cantor's Theorem)**.** Any countable linear order which is order dense (meaning $x < y$ implies there is z such that $x < z < y$), and has no first or last element is order isomorphic to $(\mathbb{Q}, <)$.

This theorem will be used more than once.

1.1. **Easy properties of $\mathbf{2}^{\mathbb{N}} = \mathbb{Z}_2^{\mathbb{N}}$.** Let $\mathbf{2} = \mathbb{Z}_2 = \{0, 1\}$ be the additive group of integers modulo 2 with the discrete topology, and let $\mathbf{2}^{\mathbb{N}}$ be its countable infinite power. Then $\mathbf{2}^{\mathbb{N}}$ under pointwise addition is an uncountable abelian compact topological group.

Fact. If A and B are countable subsets of $\mathbf{2}^{\mathbb{N}}$, then for some $p \in \mathbf{2}^{\mathbb{N}}$, $p + A$ is disjoint from B, where $p + A$ is the p-translate of A = the set $\{p + x : x \in A\}$.

Proof. (Works in any uncountable group.) The set $C = \{b - a : b \in B \text{ and } a \in A\}$ is countable as A and B are both countable, so just pick any $p \in \mathbf{2}^{\mathbb{N}} \setminus C$. □

Fact. $\mathbf{2}^{\mathbb{N}}$ is homeomorphic to the Cantor set K.

Proof. Map a binary infinite sequence $p \in \mathbf{2}^{\mathbb{N}}$ to the real number

$$\sum_{n=1}^{\infty} \frac{2p[n]}{3^n}.$$

This mapping is seen to be continuous bijection, and so a homeomorphism, as $\mathbf{2}^{\mathbb{N}}$ is compact and K is Hausdorff. □

Since for any fixed $p \in \mathbf{2}^{\mathbb{N}}$, the "translation" map $x \to p + x$ is an autohomeomorphism of $\mathbf{2}^{\mathbb{N}}$, the above two Facts imply:

Corollary 1. If E is a countable subset of the Cantor set K, then any countable dense subset of K is homeomorphic to a countable dense subset of $K \smallsetminus E$.

Theorem 2. Every T_0 space Y with a countable basis consisting of clopen sets is homeomorphic to a subset of the Cantor set.

Proof. Fix a countable basis $\{C_n : n \in \mathbb{N}\}$ of clopen sets, and let f_n be the characteristic function of C_n. Then for any n, $f_n \colon Y \to \{0,1\} = \mathbf{2}$ is continuous. Define $f \colon Y \to \mathbf{2}^{\mathbb{N}}$ by $f(x)[n] = f_n(x)$. Thus f is the unique map for which $\pi_n \circ f = f_n$ for all n, where $\pi_n \colon \mathbf{2}^{\mathbb{N}} \to \{0,1\}$ is the n-th projection map. Since the family $\langle f_n \rangle$ separates points, and also separates points-and-closed-sets, f is an embedding of Y into $\mathbf{2}^{\mathbb{N}}$. The result now follows as $\mathbf{2}^{\mathbb{N}}$ is homeomorphic to the Cantor set. □

1.2. **Order topology of R and the Cantor set.** First we review some basic facts about order topology.

Let $(X, <)$ be a linear order, and A be a subset of X.

Definition. A point $p \in X$ is an *upper limit point* of A if p is not the first element of X and for all $x \in X$, $x < p$ implies there is $a \in A$ such that $x < a < p$.

Similarly define *lower limit point*, and call a point a *two-sided limit point* of A if it is both an upper limit point and a lower limit point of A.

Definition. X is a *Dedekind completion* of A if X is order-complete (has no Dedekind gaps), and every point of $X \smallsetminus A$ is a two-sided limit point of A.

Fact (Uniqueness of Dedekind completion)**.** If X is a Dedekind completion of A, and Y is a Dedekind completion of B, and A and B are order isomorphic, then X and Y are order isomorphic.

Proof. If $f \colon A \to B$ is an order isomorphism, then for any $x \in X \smallsetminus A$, x determines a Dedekind gap (L, U) in A, and so $(f[L], f[U])$ is a Dedekind gap in B. But Y is a Dedekind completion of B, so there is a unique $y \in Y$ such that $f[L] < y < f[U]$. Set $f^*(x) = y$. The map f^* thus defined is an extension of f and an order isomorphism of X onto Y. □

Example. $\mathbb{R}$ is a Dedekind completion of $\mathbb{Q}$.

Example. The points of the Cantor set K can be divided into two disjoint classes:

(a) The countable set E of "external" points of K consists of the points 0, 1, and the endpoints of all open intervals removed in the construction of K.

(b) The points of K which are two-sided limit points of $K = K \smallsetminus E = \{x \in K :$ for all $\epsilon > 0$, there are $a, b \in K$ such that $x - \epsilon < a < x < b < x + \epsilon\}$.

Any $x \in K$ has a ternary expansion not containing the digit 1, and $x \in E$ iff this ternary expansion is eventually constant (0 or 2). It is not hard to see that for every point $x \in K \smallsetminus E$, x is a two-sided limit point of E (and x is also a two-sided limit point of $K \smallsetminus E$). Thus K is a Dedekind-completion of $E \subseteq K$. In the theorem below we will see that this can be generalized to any nowhere-dense perfect compact subset of $\mathbb{R}$.

Let X be a linear order with the order topology.

If Y is a subset of X, there are two natural topologies on Y:

(a) The relative topology on Y as a topological subspace of X, and
(b) the order topology on Y as a suborder of X.

Fact. The order topology on Y is weaker than the subspace topology.

Example. Let $X = (R, <)$ and $Y = [0, 1) \cup [2, 3]$. Then the order topology on Y is strictly weaker than the subspace topology on Y; Y with the order topology is homeomorphic to $[0, 1]$, but Y with the subspace topology is neither connected nor compact.

Under certain conditions the order topology on a subset coincides with the subspace topology:

Fact. Let X be a linear order with the order topology. For a subset Y of X, if either Y is compact in the subspace topology, or if every point of Y is a two-sided limit point of Y, then the order topology on Y coincides with the subspace topology.

Theorem. All nowhere-dense perfect compact subsets of $\mathbb{R}$ are order-isomorphic.

Proof. Let A and B be nowhere-dense perfect compact subsets of $\mathbb{R}$, and let $a_1 = \inf A$, $a_2 = \sup A$. Then $a_1, a_2 \in A$ (as A is compact), and $A \subseteq [a_1, a_2]$. Now $[a_1, a_2] \setminus A = (a_1, a_2) \setminus A$ is an open set in $\mathbb{R}$, and so it is a countable disjoint union of open intervals. Let $\mathcal{S}$ be the family of these open intervals, i.e. the components of $[a_1, a_2] \setminus A$. If $I, J \in \mathcal{S}$, we say $I < J$ if some (and so any) point of I is less than some (and so any) point of J. Thus the ordering of the reals induce an ordering of $\mathcal{S}$. It is easy to see that this ordering on $\mathcal{S}$ is order-dense (because A is nowhere dense and perfect), and without first or last point. Moreover $\mathcal{S}$ is countable. Let E be the set of end-points of the intervals of $\mathcal{S}$ together with a_1 and a_2.

Similarly take $\mathcal{T}$ to be the component intervals of $[b_1, b_2] \setminus B$, where $b_1 = \inf B$ and $b_2 = \sup B$, and order $\mathcal{T}$ naturally to get another countable order-dense set without first or last point. Let F be the set of end-points of the intervals of $\mathcal{T}$ together with b_1 and b_2.

By Cantor's theorem there is an order preserving bijection from $\mathcal{S}$ onto $\mathcal{T}$. This bijection naturally induces an order preserving bijection between E and F, thus E and F are order-isomorphic. Now note that A is a Dedekind completion of E, and B is a Dedekind completion of F, so A and B are order-isomorphic. □

Since the Cantor set is a nowhere-dense perfect compact subset of $\mathbb{R}$, we have:

Corollary. Any nowhere-dense perfect compact subset of $\mathbb{R}$ is order isomorphic to the Cantor set.

Since for any compact subset of $\mathbb{R}$, the order topology coincides with the subspace topology, we have:

Corollary 3. Any nowhere-dense perfect compact subset of $\mathbb{R}$ is homeomorphic to the Cantor set.

Note A: The proof of the theorem shows that the order type of the Cantor set (and thus of any nowhere-dense perfect compact subset of $\mathbb{R}$) can be characterized as the Dedekind completion of $1 + 2\eta + 1$, where η is the order type of $(\mathbb{Q}, <)$.

Note B: Brouwer's characterization of the Cantor space as the unique zero-dimensional compact perfect metrizable space can be derived from this corollary.

1.3. **The Proof.** Let X be any countable metrizable space without isolated points. Then X has a countable basis consisting of clopen sets. To see this, let $p \in X$. Put $S(p,r) = \{x : d(p,x) = r\}$. Then for any $\epsilon > 0$, $S(p,r)$ is empty for at least one positive real $r < \epsilon$, and for this r, the open ball of radius r centered at p is clopen.

Hence by Theorem 2 (Section 1.1):

X is homeomorphic to a dense-in-itself subset of the Cantor set.

Since the closure of a dense-in-itself subset of the Cantor set is a nowhere-dense perfect compact subset of R, we get:

X densely embeds in a nowhere-dense perfect compact subset of $\mathbb{R}$.

By Corollary 3 (Section 1.2):

X densely embeds in the Cantor set K.

By Corollary 1 (Section 1.1):

X is homeomorphic to a countable dense subset D of $K \setminus E$,

where E is the countable set of "external endpoints" of the Cantor set.

Since D is a dense subset of $K \setminus E$, every point of D is a two-sided limit point of D, so the subspace topology on D coincides with the order topology on D. Hence:

X is homeomorphic to $(D, <)$ with order topology.

Again because D is a countable dense subset of $K \setminus E$, the linear order $(D, <)$ is countable, order-dense, and without endpoints. So by Cantor's theorem (second application!):

$(D, <)$ is order isomorphic to $(Q, <)$.

Finally it follows:

X is homeomorphic to $(Q, <)$ with order topology.

2. Proof using the Alexandrov-Urysohn theorem

Definition. A topological space is *nowhere compact* if every compact subset has empty interior.

The following fact is proved by a routine elementary topological argument:

Fact. If X is a subspace of a Hausdorff space Y, and both X and $Y \setminus X$ are dense in Y, then X is nowhere compact.

Theorem (Alexandrov-Urysohn)**.** A zero-dimensional nowhere compact separable complete metric space is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$, the subspace of irrationals in $\mathbb{R}$.

We now give a proof of Sierpinski's theorem using the above theorem of Alexandrov and Urysohn (which we do not prove here).

Proof. Let $X = \{x_m : m \in \mathbb{N}\}$ be a countable metric space without isolated points. We regard X as a subset of its metric completion (X^*, d).

For each $m, n \in \mathbb{N}$ there is $r_{m,n} > 0$ with $r_{m,n} < 1/n$ and $S(x_m, r_{m,n}) \cap X = \emptyset$, where $S(p,r) = \{x \in X^* : d(p,x) = r\}$. Let $H = X^* \setminus \cup_{m,n \in \mathbb{N}} S(x_m, r_{m,n})$. Then H is zero-dimensional by construction. Also H is a $\mathcal{G}_\delta$ subset of X^* containing X, so H is completely metrizable (recall that a subset of a complete metric space is completely metrizable iff it is a $\mathcal{G}_\delta$), and of course H is separable and without isolated points. Thus X is a meager subset of H and by the Baire category theorem, $H \setminus X$ is dense in H. Now choose a countable dense subset D of $H \setminus X$, and put $Y = H \setminus D$. Again, Y is a $\mathcal{G}_\delta$ subset of H containing X, so Y is a separable completely metrizable zero-dimensional space containing X. Moreover, both Y and $H \setminus Y = D$ are dense in H, so Y is nowhere compact. Hence by the Alexandrov-Urysohn theorem, Y is homeomorphic to the irrationals.

It follows that X is homeomorphic to a dense subspace of the irrationals, and hence to a dense subspace of $\mathbb{R}$.

But a countable dense set in $\mathbb{R}$ is homeomorphic to $\mathbb{Q}$ by Cantor's Theorem. □

3. A direct back-and-forth argument

Theorem. Let X and Y be T_0 spaces without isolated points and each possessing a countable basis consisting of clopen sets. Let A and B be a countable dense subsets of X and Y respectively. Then there is a bijection $f\colon A \to B$ which is a homeomorphism from A onto B. If in addition Y is compact, then f extends uniquely to a relative homeomorphism f^* from X into Y. If X is also compact then f^* is a homeomorphism of X onto Y.

Proof. Assume that $\mathcal{S}$ is a countable algebra (field) of clopen subsets of X which forms a basis for the topology of X. Similarly, let $\mathcal{T}$ be a countable algebra (field) of clopen subsets of Y which forms a basis for the topology of Y.

By a *partition* P of a set E we mean a collection of non-empty disjoint subsets of E whose union is E. If P is a partition of E, and $x \in E$, then $P[x]$ denotes the unique member of P containing x. A subset C of E is a *choice set* for the partition P if $P = \{P[x] : x \in C\}$, and $P[x] \neq P[y]$ for any distinct $x, y \in C$.

By a *condition* we mean a triple (P, Q, f) satisfying:

(a) $P \subseteq \mathcal{S}$ is a finite partition of X (by clopen sets from $\mathcal{S}$),
(b) $Q \subseteq \mathcal{T}$ is a finite partition of Y (by clopen sets from $\mathcal{T}$),
(c) f is a finite function such that $\mathrm{dom}(f) \subseteq A$ and $\mathrm{ran}(f) \subseteq B$,
(d) $\mathrm{dom}(f)$ is a choice set for P, and
(e) $\mathrm{ran}(f)$ is a choice set for Q.

We say that a condition (P_2, Q_2, f_2) *extends* a condition (P_1, Q_1, f_1) if P_2 refines P_1, Q_2 refines Q_1, $f_2 \supseteq f_1$, and for any $x, a \in \mathrm{dom}(f_2)$, $x \in P_1[a]$ iff $f_2[x] \in Q_1[f_2(a)]$.

It is easily seen that this relation is reflexive, antisymmetric, and transitive.

Lemma. Given a condition (P_1, Q_1, f_1) and $a \in A$ (resp. $b \in B$), there is a condition (P_2, Q_2, f_2) extending (P_1, Q_1, f_1) such that $a \in \mathrm{dom}(f_2)$ (resp. $b \in \mathrm{ran}(f_2)$). Given a condition (P_1, Q_1, f_1) and a set $S \in \mathcal{S}$ (resp. $T \in \mathcal{T}$) there is a condition (P_2, Q_2, f_2) extending (P_1, Q_1, f_1) such that S is a union of sets in P_2 (resp. T is a union of sets in Q_2).

The proof of the lemma is left as an exercise.

Now enumerate $\mathcal{S} = \{S_n : n \in \mathbb{N}\}$, $\mathcal{T} = \{T_n : n \in \mathbb{N}\}$, $A = \{a_n : n \in \mathbb{N}\}$, and $B = \{b_n : n \in \mathbb{N}\}$. Let $P_1 = \{X\}$, $Q_1 = \{Y\}$, and $f_1 = \{\langle a, b\rangle\}$, where $a \in A$ and $b \in B$ are fixed arbitrarily. Given a condition (P_n, Q_n, f_n), use the lemma to inductively choose a condition $(P_{n+1}, Q_{n+1}, f_{n+1})$ extending (P_n, Q_n, f_n) such that S_n is a union of sets in P_{n+1}, T_n is a union of sets in Q_{n+1}, $a_n \in \mathrm{dom}(f_{n+1})$, and $b_n \in \mathrm{ran}(f_{n+1})$. Finally let $f = \cup_n f_n$.

By construction, $f\colon A \to B$ is a bijection. Moreover, for any $S \in \mathcal{S}$ there is a unique $T \in \mathcal{T}$ such that for any $a \in A$, $a \in S$ iff $f(a) \in T$, and similarly for any $T \in \mathcal{T}$ there is $S \in \mathcal{S}$ such that for any $a \in A$, $a \in S$ iff $f(a) \in T$. This defines a bijection $H\colon \mathcal{S} \to \mathcal{T}$ with the property that for any $S \in \mathcal{S}$ and $a \in A$, $a \in S$ iff $f(a) \in H(S)$. (H can be seen to be a set-algebra isomorphism.)

It follows that f is a homeomorphism of A onto B.

If Y is compact, then given any $x \in X$, let $\mathcal{V}_x = \{H(S) : x \in S,\ S \in \mathcal{S}\}$. Then $\mathcal{V}_x$ is a filter of clopen subsets of Y with the property that for any $T \in \mathcal{T}$, either $T \in \mathcal{V}_x$ or $Y \setminus T \in \mathcal{V}_x$. By this property and compactness of Y, $\bigcap \mathcal{V}_x$ is a singleton $\{y\}$. Set $f^*(x) = y$. Then $f^*\colon X \to Y$ is an embedding.

If X is also compact, then the image of f^* is a compact subset of Y containing the dense set B, so f^* must be onto, and hence a homeomorphism of X onto Y. $\square$

The theorem immediately implies Brouwer's characterization of the Cantor set, and more:

Corollary (Brouwer)**.** Any two second countable compact zero-dimensional spaces without isolated points are homeomorphic. In fact, they are countable dense homogeneous, meaning that given countable dense subsets of the two spaces, a homeomorphism can be found which maps one dense subset onto the other.

Corollary (Brouwer)**.** The Cantor set is the topologically unique second countable compact zero-dimensional space without isolated points. Moreover, it is countable dense homogeneous.

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